

# The onset of thermo-bioconvection in a shallow fluid saturated porous layer heated from below in a suspension of oxytactic microorganisms

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## Abstract

The aim of this paper is to present a continuum model for thermo-bioconvection of oxytactic bacteria in a porous medium and investigate the combined effects of microorganisms' upswimming and heating from below on the stability of bioconvection in a horizontal layer filled with a fluid saturated porous medium. Different from traditional bioconvection, thermo-bioconvection has two destabilizing mechanisms that contribute to creating the unstable density stratification. This problem may be relevant to a number of geophysical applications, such as the investigation of the dynamics of oxytactic species of thermophiles (heat loving microorganisms) living in hot springs, microbial-enhanced oil recovery, and modeling oil- and gas-bearing sedimentary basins. The utilization of the Galerkin method to solve a linear stability problem leads to a correlation between the critical value of the bioconvection Rayleigh number and the traditional "thermal" Rayleigh number.

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## 1. Introduction

Different from traditional bioconvection, which is caused by upwardly swimming microorganisms whose average density is slightly larger than that of water (Pedley and Kessler [1], Hillesdon and Pedley [2], Hillesdon et al. [3], Metcalfe and Pedley [4,5]), thermo-bioconvection is caused by two factors: the upswimming of microorganisms and the temperature gradient across the fluid. Thermo-bioconvection may be important for understanding fluid motion in hot springs populated by motile thermophiles (heat loving microorganisms).

This paper continues the research started in [6] and extends the analysis of stability of thermo-bioconvection in a suspension of oxytactic bacteria to a horizontal layer filled with a fluid saturated porous medium. This problem may be relevant to modeling oil and gas-bearing sedimentary basins (Polyansky et al. [7]). Another example is microbial-enhanced oil recovery, when microorganisms and nutrients are injected in oil-bearing layers to correct permeability variation (Stewart and Fogler [8], Kim and Fogler [9]).

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Although Amiri and Vafai [10] and Khaled and Vafai [11] have demonstrated the importance of non-Darcian effects for modeling flow in porous media and in biological tissues, the momentum equation utilized in this paper is based on the Darcy law. This is done in order to minimize the number of parameters involved in the model.

In designing a model of bioconvection in a porous medium, it is assumed that the porous matrix does not absorb microorganisms and that the pore sizes are significantly larger than the microorganisms; therefore, oxytactic behavior of microorganisms is not affected by the presence of the porous matrix. It is also assumed that heating from below is sufficiently weak so that it does not kill microorganisms and does not affect their oxytactic behavior. The most interesting outcome of this paper is a correlation between two Rayleigh numbers: the traditional, “thermal”, Rayleigh number, which is associated with the temperature gradient in the fluid layer, and the bioconvection Rayleigh number, which is associated with the density difference induced by the upswimming of microorganisms.

## 2. Governing equations

Fig. 1 displays a schematic diagram of the problem. A horizontal fluid saturated porous layer of depth  $H$  confined between  $\tilde{z} = 0$  (the top surface) and  $\tilde{z} = H$  (the bottom surface) and unbounded in the  $\tilde{x}$  and  $\tilde{y}$  directions is considered, where  $\tilde{z}$  is positively oriented downward. The layer is heated from below and both top and bottom surfaces are assumed to be at uniform temperatures,  $\tilde{T}_0$  and  $\tilde{T}_0 + \Delta T$ , respectively. Heating from below is expected to enhance instability by adding a second destabilizing mechanism to that already existing in a suspension of oxytactic bacteria: bacteria, which are heavier than water, tend to migrate to the regions just below the top surface where the oxygen concentration is the highest, thus creating an unstable density stratification with the maximum density at the top surface.

The model presented here is based on a continuum model of a suspension of oxytactic microorganisms developed in Hillesdon and Pedley [2]. This model is supplemented by an energy equation and a buoyancy term in the momentum equation that results from the temperature variation across the layer. The Boussinesq approximation is utilized and the suspension is assumed dilute. The inertia terms are neglected due to the small flow velocity associated with bioconvection (Pedley et al. [12]). Governing equations for a porous medium are obtained by volume averaging these equations. The volume averaging procedure is described in detail in Whitaker [13]. This procedure results in the replacement of the Laplacian viscous terms with the Darcian terms that describe viscous resistance in a porous medium (Nield and Bejan [14]). The resulting governing equations are:

*Momentum equation*

$$c_a \rho_w \frac{\partial \tilde{\mathbf{U}}}{\partial \tilde{t}} = -\tilde{\nabla} \tilde{p} - \frac{\mu}{K} \tilde{\mathbf{U}} + \tilde{n} \theta \Delta \rho \mathbf{g} - \rho_w \hat{\beta} (\tilde{T} - \tilde{T}_0) \mathbf{g}, \quad (1)$$

where  $c_a$  is the acceleration coefficient introduced by Nield and Bejan [14], it depends on the geometry of the porous medium and is determined mainly by the nature of the pore channels of the largest cross sections (as explained in [14], in narrow pore channels the transients decay more rapidly);  $\mathbf{g}$  is the gravity vector;  $K$  is the permeability of the porous medium;  $\tilde{n}$  is the number density of motile microorganisms;  $\tilde{p}$  is the excess pressure (above hydrostatic);  $\tilde{t}$  is the time;  $\tilde{\mathbf{U}} = (\tilde{u}, \tilde{v}, \tilde{w})$  is the fluid filtration velocity;  $\hat{\beta}$  is the volume expansion coefficient of water at constant pressure;

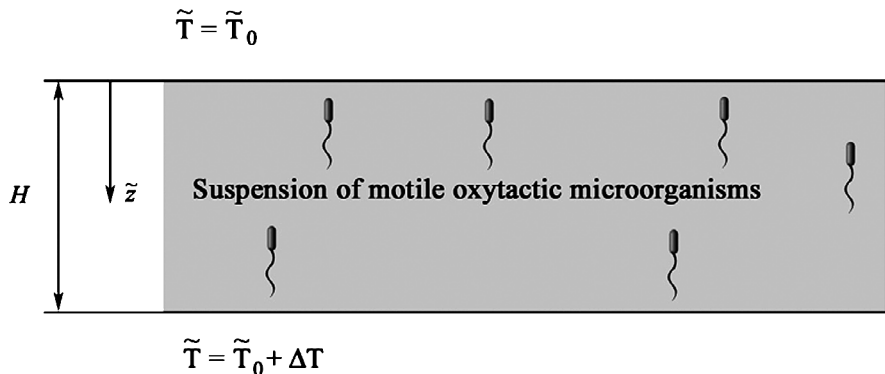


Fig. 1. Schematic diagram of the problem.

$\Delta\rho = \rho_{\text{cell}} - \rho_w$  is the density difference between cells and water;  $\theta$  is the average volume of a microorganism;  $\mu$  is the dynamic viscosity, assumed to be approximately the same as that of water; and  $\rho_w$  is the density of water; tildes denote dimensional variables.

*Continuity equation*

$$\tilde{\nabla} \cdot \tilde{\mathbf{U}} = 0. \quad (2)$$

*Cell conservation*

$$\varphi \frac{\partial \tilde{n}}{\partial \tilde{t}} = -\text{div}(\tilde{\mathbf{j}}), \quad (3)$$

where

$$\tilde{\mathbf{j}} = \tilde{n}\tilde{\mathbf{U}} + \tilde{n}\tilde{\mathbf{V}} - D_n\tilde{\nabla}\tilde{n} \quad (4)$$

is the flux of microorganisms. The terms on the right-hand side of Eq. (4) represent the flux of microorganisms due to the macroscopic motion of the fluid, the directional swimming of microorganisms up the oxygen gradients, and a diffusive process that models all random motions of microorganisms, respectively.  $D_n$  in Eq. (4) is the diffusivity of the microorganisms.

The average directional swimming velocity of a microorganism can be approximated as (Hillesdon and Pedley [2]):

$$\tilde{\mathbf{V}} = bW_c\hat{H}(C)\tilde{\nabla}C, \quad (5)$$

where  $b$  is the chemotaxis constant [m] and  $W_c$  is the maximum cell swimming speed [m/s] (the product  $bW_c$  is assumed to be constant). The dimensionless oxygen concentration,  $C$ , in Eq. (5) is defined as:

$$C = \frac{\tilde{C} - \tilde{C}_{\min}}{\tilde{C}_0 - \tilde{C}_{\min}}, \quad (6)$$

where  $\tilde{C}$  is the dimensional oxygen concentration,  $\tilde{C}_0$  is the free-surface oxygen concentration, and  $\tilde{C}_{\min}$  is the minimum oxygen concentration that microorganisms need in order to be active. Since for the shallow layer  $C$  is greater than zero throughout the layer thickness, the Heaviside step function,  $\hat{H}(C)$ , in Eq. (5) is equal to unity.

*Thermal energy equation*

$$c_p\rho_w\left(\frac{\partial \tilde{T}}{\partial \tilde{t}} + \tilde{\mathbf{U}} \cdot \tilde{\nabla}\tilde{T}\right) = k\tilde{\nabla}^2\tilde{T}, \quad (7)$$

where  $c_p$  is the specific heat of water,  $k$  is the effective thermal conductivity of the porous medium,  $\tilde{T}$  is the temperature, and  $\tilde{\nabla}^2$  is the Laplacian operator.

*Oxygen conservation*

$$\varphi \frac{\partial C}{\partial \tilde{t}} + \tilde{\mathbf{U}} \cdot \tilde{\nabla}C = D_C\tilde{\nabla}^2C - \frac{\gamma\tilde{n}}{\Delta\tilde{C}}, \quad (8)$$

where  $D_C$  is the diffusivity of oxygen,  $-\gamma\tilde{n}/\Delta\tilde{C}$  describes the consumption of oxygen by the microorganisms in the fluid, and  $\Delta\tilde{C}$  equals  $\tilde{C}_0 - \tilde{C}_{\min}$ .

Porosity,  $\varphi$ , is involved in the term on the left-hand side of Eqs. (3) and (8) because in the porous medium, the concentrations of cells and oxygen (unlike the heat) are advected/convected with the intrinsic velocity (not the Darcy filtration velocity) since the cells and oxygen cannot pass through the solid phase (Nield [15]). An extra factor  $\varphi$  has been incorporated into  $\gamma$  and the effective transport coefficients for the porous medium,  $D_n$ ,  $D_C$ , and  $bW_c$ , in Eqs. (4), (5), and (8).

### 3. Boundary conditions

In accordance with Fig. 1, the following boundary conditions are imposed at the top surface:

$$\text{At } \tilde{z} = 0: \tilde{w} = 0, \quad C = 1, \quad \tilde{T} = \tilde{T}_0, \quad \tilde{\mathbf{j}} \cdot \hat{\mathbf{k}} = 0, \quad (9)$$

where  $\hat{\mathbf{k}}$  is the vertically-downward unit vector.

The last equation in (9) can be recast as:

$$\tilde{n}bW_c \frac{\partial C}{\partial \tilde{z}} - D_n \frac{\partial \tilde{n}}{\partial \tilde{z}} = 0. \quad (10)$$

At the bottom of the layer (assumed to be rigid) the following conditions are satisfied:

$$\text{At } \tilde{z} = H: \tilde{w} = 0, \quad \frac{\partial C}{\partial \tilde{z}} = 0, \quad \tilde{T} = \tilde{T}_0 + \Delta T, \quad \tilde{\mathbf{j}} \cdot \hat{\mathbf{k}} = 0, \quad (11)$$

where  $\Delta T$  is positive when heating from below is considered.

The last equation in (11), accounting for  $\partial C / \partial \tilde{z} = 0$ , can be recast as:

$$\frac{\partial \tilde{n}}{\partial \tilde{z}} = 0. \quad (12)$$

#### 4. Dimensionless equations

Dimensionless variables and operators are introduced as follows:

$$n = \tilde{n} / \tilde{n}_0, \quad [x, y, z] = [\tilde{x}, \tilde{y}, \tilde{z}] / H, \\ t = \left( \frac{D_n}{H^2} \right) \tilde{t}, \quad \mathbf{U} = \left( \frac{H}{D_n} \right) \tilde{\mathbf{U}}, \quad p = \left( \frac{H^2}{\mu D_n} \right) \tilde{p}, \quad T = \frac{\tilde{T} - \tilde{T}_0}{\Delta T}, \quad \nabla^2 = H^2 \tilde{\nabla}^2, \quad \nabla = H \tilde{\nabla}, \quad (13)$$

where  $\tilde{n}_0$  is the average number density of the microorganisms (number density of the microorganisms in a well-stirred suspension) and  $\mathbf{U} = (u, v, w)$  is the dimensionless fluid filtration velocity.

Dimensionless constants are defined as

$$\delta = \frac{D_C}{D_n}, \quad Pe = \frac{bW_c}{D_n}, \quad \beta = (\gamma \tilde{n}_0 H^2) / (D_C \Delta \tilde{C}), \quad Le = \frac{D_C c_p \rho_w}{k}, \\ Sc = \frac{\mu}{\rho_w D_n}, \quad Ra = \frac{g \beta \Delta T H^3 \rho_w^2 c_p}{\mu k}, \quad Rb = \frac{\Delta \rho \theta \tilde{n}_0 g}{\mu D_n} H^3, \quad Da = \frac{K}{H^2}, \quad (14)$$

where  $Da$  is the Darcy number,  $Le$  is the Lewis number,  $Ra$  is the traditional Rayleigh number,  $Rb$  is the bioconvection Rayleigh number, and  $Sc$  is the Schmidt number.  $\beta$  characterizes the ratio of the rate of oxygen consumption to the rate of oxygen diffusion, it can be regarded as a depth parameter.  $Pe$  can be regarded as a ratio of two characteristic velocities; one due to oxytactic swimming and the other due to random, diffusive swimming (Hillesdon and Pedley [2]).

The dimensionless governing equations can be presented as:

$$\frac{c_a}{Sc} \frac{\partial \mathbf{U}}{\partial t} = -\nabla p - \frac{1}{Da} \mathbf{U} + \hat{\mathbf{k}} \left[ Rb n - Ra \frac{\delta}{Le} T \right], \quad (15)$$

$$\nabla \cdot \mathbf{U} = 0, \quad (16)$$

$$\varphi \frac{\partial n}{\partial t} = -\text{div}(\mathbf{j}), \quad (17)$$

where  $\mathbf{j} = \tilde{\mathbf{j}} H / (D_n \tilde{n}_0) = n \mathbf{U} + n \mathbf{V} - \nabla n$  and  $\mathbf{V} = \tilde{\mathbf{V}} H / D_n$

$$\frac{\partial T}{\partial t} + \mathbf{U} \cdot \nabla T = \frac{\delta}{Le} \nabla^2 T, \quad (18)$$

$$\varphi \frac{\partial C}{\partial t} + \mathbf{U} \cdot \nabla C = \delta \nabla^2 C - \beta \delta n. \quad (19)$$

Dimensionless boundary conditions can be presented as follows. At the top surface the following boundary conditions are satisfied:

$$\text{At } z = 0: w = 0, \quad C = 1, \quad T = 0, \quad Pen \frac{\partial C}{\partial z} - \frac{\partial n}{\partial z} = 0. \quad (20)$$

At the bottom surface the following dimensionless boundary conditions are satisfied:

$$\text{At } z = 1: w = 0, \quad \frac{\partial C}{\partial z} = 0, \quad T = 1, \quad \frac{\partial n}{\partial z} = 0. \quad (21)$$

## 5. Basic state

In the basic state the fluid is motionless and cell and oxygen concentrations change in the  $z$ -direction only. The solutions for  $C_b(z)$  and  $n_b(z)$  follow from Hillesdon and Pedley [2] and the solution for  $T_b(z)$  and  $p_b(z)$  are obtained by integrating Eqs. (18) and (15), respectively, with boundary conditions (20) and (21):

$$C_b(z) = 1 - \frac{2}{Pe} \ln \left( \frac{\cos\{A_1(1-z)/2\}}{\cos\{A_1/2\}} \right), \quad (22)$$

$$n_b(z) = \frac{A_1^2}{2Pe\beta} \sec^2 \left( \frac{A_1}{2}(1-z) \right), \quad (23)$$

$$T_b(z) = z \quad (24)$$

$$p_b(z) = p_0 + Rb \frac{A_1}{Pe\beta} \left[ \tan \left( \frac{A_1}{2} \right) - \tan \left( \frac{A_1}{2}(1-z) \right) \right] - Ra \frac{\delta}{Le} \frac{z^2}{2}, \quad (25)$$

where  $p_0 = H^2 \tilde{p}_0 / (\mu D_n)$  is the dimensionless pressure at the top surface,  $\tilde{p}_0$  is the pressure at the top surface, and the constant  $A_1$  is found from the transcendental equation

$$\tan \left( \frac{A_1}{2} \right) = \frac{Pe\beta}{A_1}. \quad (26)$$

The solution for the basic state given by Eqs. (22)–(25) is valid as long as the dimensionless oxygen concentration is positive throughout the chamber. In [2] it is shown that this condition holds as long as

$$\beta \leq \frac{2\phi}{Pe} \tan^{-1} \phi, \quad (27)$$

where

$$\phi^2 = \exp(Pe) - 1. \quad (28)$$

Since  $\beta$  is proportional to  $H^2$ , Eq. (27) determines the largest layer depth for which the layer can be treated as shallow.

## 6. Linear stability analysis

The perturbations are introduced as follows:

$$n(t, x, y, z) = n_b(z) + \varepsilon n^*(t, x, y, z), \quad (29)$$

$$\mathbf{U}(t, x, y, z) = \varepsilon \mathbf{U}^*(t, x, y, z), \quad (30)$$

$$p(t, x, y, z) = p_b(z) + \varepsilon p^*(t, x, y, z), \quad (31)$$

$$T(t, x, y, z) = T_b(z) + \varepsilon T^*(t, x, y, z), \quad (32)$$

$$C(t, x, y, z) = C_b(z) + \varepsilon C^*(t, x, y, z), \quad (33)$$

where an asterisk denotes a perturbation quantity,  $\mathbf{U}^* = (u^*, v^*, w^*)$ , and  $\varepsilon$  is the small perturbation amplitude. Substituting Eqs. (29)–(33) into Eqs. (15)–(19) and linearizing results in the following equations for perturbations:

$$\frac{c_a}{Sc} \frac{\partial \mathbf{U}^*}{\partial t} = -\nabla p^* - \frac{1}{Da} \mathbf{U}^* + \hat{\mathbf{k}} \left[ Rb n^* - Ra \frac{\delta}{Le} T^* \right], \quad (34)$$

$$\nabla \cdot \mathbf{U}^* = 0, \quad (35)$$

$$\varphi \frac{\partial n^*}{\partial t} + w^* \frac{\partial n_b(z)}{\partial z} + Pe \frac{\partial C^*}{\partial z} \frac{\partial n_b(z)}{\partial z} + Pe \frac{\partial C_b(z)}{\partial z} \frac{\partial n^*}{\partial z} + Pe n^*(z) \frac{\partial^2 C_b(z)}{\partial z^2} + Pe n_b(z) \nabla^2 C^* = \nabla^2 n^*, \quad (36)$$

$$\frac{\partial T^*}{\partial t} + w^* = \frac{\delta}{Le} \nabla^2 T^*, \quad (37)$$

$$\varphi \frac{\partial C^*}{\partial t} + w^* \frac{\partial C_b(z)}{\partial z} = \delta \nabla^2 C^* - \beta \delta n^*. \quad (38)$$

Elimination of  $u^*$ ,  $v^*$ , and  $p^*$  from Eqs. (34) and (35) results in:

$$\begin{aligned} \frac{c_a}{Sc} \frac{\partial}{\partial t} \left( \frac{\partial^2 w^*}{\partial x^2} + \frac{\partial^2 w^*}{\partial y^2} + \frac{\partial^2 w^*}{\partial z^2} \right) = & Rb \left( \frac{\partial^2 n^*}{\partial x^2} + \frac{\partial^2 n^*}{\partial y^2} \right) - Ra \frac{\delta}{Le} \left( \frac{\partial^2 T^*}{\partial x^2} + \frac{\partial^2 T^*}{\partial y^2} \right) \\ & - \frac{1}{Da} \left( \frac{\partial^2 w^*}{\partial x^2} + \frac{\partial^2 w^*}{\partial y^2} + \frac{\partial^2 w^*}{\partial z^2} \right). \end{aligned} \quad (39)$$

A normal mode expansion is introduced in the form:

$$[w^*, n^*, T^*, C^*] = [W(z), N(z), \Theta(z), \Xi(z)] f(x, y) \exp(\sigma t). \quad (40)$$

The function  $f(x, y)$  satisfies the following equation:

$$\frac{\partial^2 f}{\partial x^2} + \frac{\partial^2 f}{\partial y^2} = -a^2 f, \quad (41)$$

where  $a$  is the horizontal wavenumber (used as a separation constant).

Substituting Eq. (40) into Eqs. (36)–(39), and accounting for Eq. (41), the following equations for the amplitudes  $W$ ,  $\Theta$ ,  $N$ , and  $\Xi$  are obtained:

$$a^2 Da Le Rb Sc N - a^2 Le (Sc + c_a Da \sigma) W - a^2 Da Ra Sc \delta \Theta + Le (Sc + c_a Da \sigma) W'' = 0, \quad (42)$$

$$Le W + (a^2 \delta + Le \sigma) \Theta - \delta \Theta'' = 0, \quad (43)$$

$$\begin{aligned} 2A_1 Pe \beta \tan \left[ \frac{1}{2} A_1 (z-1) \right] N'(z) + A_1^3 \sec^2 \left[ \frac{1}{2} A_1 (z-1) \right] \tan \left[ \frac{1}{2} A_1 (z-1) \right] (W + Pe \Xi') \\ + 2Pe \beta ((a^2 + \sigma \varphi) N - N'') + A_1^2 Pe \sec^2 \left[ \frac{1}{2} A_1 (z-1) \right] (\beta N - a^2 \Xi + \Xi'') = 0, \end{aligned} \quad (44)$$

$$Pe \beta \delta N - A_1 \tan \left[ \frac{1}{2} A_1 (1-z) \right] W + Pe (a^2 \delta + \sigma \varphi) \Xi - Pe \delta \Xi'' = 0. \quad (45)$$

Eqs. (42)–(45) must be solved subject to the following boundary conditions:

$$\text{At } z=0: W=0, \quad \Xi=0, \quad \Theta=0, \quad Pe \left( n_b|_{z=0} \frac{d\Xi}{dz} + \frac{dC_b}{dz} \Big|_{z=0} N \right) - \frac{\partial N}{\partial z} = 0, \quad (46)$$

$$\text{At } z=1: W=0, \quad \frac{d\Xi}{dz}=0, \quad \Theta=0, \quad \frac{dN}{dz}=0. \quad (47)$$

The utilization of the principal of exchange of stabilities (Chandrasekhar [16]) is warranted for a fluid layer heated from below. Although there are two agencies affecting the density in this case (oxytactic upswimming of microorganisms and heating from below), both of these agencies are destabilizing; therefore, overstability in this system is not possible unless there is cooling of the fluid layer from below. (The case of cooling from below, when oscillatory instability is possible, requires a separate analysis.) Therefore,  $\sigma$  is set to zero in Eqs. (42)–(45). This makes these equations independent of the Schmidt number,  $Sc$ . Also, by letting  $\Xi \rightarrow \beta \bar{\Xi}$ , the system (42)–(47) depends on the product  $\varpi = Pe \beta$  rather than  $Pe$  and  $\beta$  individually. Finally, by letting  $\Theta \rightarrow Le \bar{\Theta}$ , the system (42)–(47) becomes independent of  $Le$ . Also, new Raleigh–Darcy numbers (traditional and bioconvection) can be introduced as:

$$\widehat{Ra} = Da Ra \quad \text{and} \quad \widehat{Rb} = Da Rb. \quad (48)$$

Rescaling  $\Xi$  and  $\Theta$ , Eqs. (42)–(45) are recast as:

$$a^2 \widehat{Rb} N - a^2 W - a^2 \widehat{Ra} \delta \bar{\Theta} + W'' = 0, \quad (49)$$

$$W + a^2 \delta \bar{\Theta} - \delta \bar{\Theta}'' = 0, \quad (50)$$

$$2A_1\varpi \tan\left[\frac{1}{2}A_1(z-1)\right]N'(z)+A_1^3\sec^2\left[\frac{1}{2}A_1(z-1)\right]\tan\left[\frac{1}{2}A_1(z-1)\right](W+\varpi\overline{\mathcal{E}}') \\ +2\varpi(a^2N-N'')+A_1^2\varpi\sec^2\left[\frac{1}{2}A_1(z-1)\right](N-a^2\mathcal{E}+\mathcal{E}'')=0, \quad (51)$$

$$\varpi \delta N - A_1 \tan \left[ \frac{1}{2} A_1 (1 - z) \right] W + \varpi a^2 \delta \bar{\Xi} - \varpi \delta \bar{\Xi}'' = 0. \quad (52)$$

Eqs. (49)–(52) must be solved subject to the following boundary conditions:

$$\text{At } z=0: W=0, \quad \bar{E}=0, \quad \bar{\Theta}=0, \quad \varpi \left( n_b|_{z=0} \frac{d\bar{E}}{dz} + \frac{1}{\beta} \frac{dC_b}{dz} \right) \bigg|_{z=0} N - \frac{\partial N}{\partial z} = 0, \quad (53)$$

$$\text{At } z = 1: W = 0, \quad \frac{d\bar{\mathcal{E}}}{dz} = 0, \quad \bar{\Theta} = 0, \quad \frac{dN}{dz} = 0. \quad (54)$$

It should be noted that the term  $\beta^{-1} dC_b/dz$  in the last equation of (53) depends on  $\varpi$ , but does not depend on  $Pe$  and  $\beta$  individually (cf. Eq. (22)).

For the solution of this system, a single term Galerkin method is employed. Suitable trial functions, which satisfy the boundary conditions given by Eqs. (53) and (54), are:

$$W_1 = z - z^2, \quad (55)$$

$$\bar{\Theta}_1 = z - z^2, \quad (56)$$

$$N_1 = 1 + \alpha \left( z - \frac{1}{2} z^2 \right), \quad (57)$$

$$\overline{\mathcal{E}}_1 = z - \frac{1}{2}z^2, \quad (58)$$

where

$$\alpha = \frac{A_1(A_1 - \sin A_1)}{1 + \cos A_1}. \quad (59)$$

The utilization of a standard Galerkin procedure (Finlayson [17]), which involves substituting the trial functions given by Eqs. (55)–(58) into Eqs. (49)–(52), calculating the residuals, and making the residuals orthogonal to the relevant trial functions, results in the following equation for the critical bioconvection Rayleigh number:

$$\begin{aligned} \widehat{Rb}_{\text{cr}} = & \min_a \left\{ 16\varpi \delta \left[ (10 + a^2)^2 - a^2 \widehat{Ra} \right] \left\{ - \frac{(2I_3 + a^2 A_1^2 I_4 \varpi)(5 + 2A_1^2 + 5 \cos(A_1) - 2A_1 \sin(A_1))}{1 + \cos(A_1)} \right. \right. \\ & + 2(5 + 2a^2) \left[ I_2 + \frac{1}{60} a^2 \varpi \sec^4 \left( \frac{A_1}{2} \right) (45 + 22A_1^2 + 4A_1^4 + 20(3 + A_1^2) \cos(A_1) \right. \\ & \left. \left. + (15 - 2A_1^2) \cos(2A_1) - 4A_1(5 + 2A_1^2 + 5 \cos(A_1)) \sin(A_1) \right) \right] \left. \right\} \\ & / \left\{ a^2 (10 + a^2) [15I_1(2I_3 + a^2 A_1^2 I_4 \varpi) - 4(5 + 2a^2)I_5 \varpi \delta] \right. \\ & \left. \times \left( 40 + 7A_1 \sec^2 \left( \frac{A_1}{2} \right) (A_1 - \sin(A_1)) \right) \right\} \end{aligned} \quad (60)$$

where the integrals  $I_1$ – $I_5$  (all of which are functions of  $\varpi$  only) are given in Appendix A.

## 7. Validation

For the case of  $\widehat{Rb} = 0$  (the microorganisms are buoyancy-neutral), Eq. (60) predicts that  $\widehat{Ra}_{\text{cr}} = 40$  at  $a_{\text{cr}} = 3.16$ . This is within 1.27% of the exact result of  $\widehat{Ra}_{\text{cr}} = 39.49$  and  $a_{\text{cr}} = 3.14$  that follows from Table 6.1 of Nield and Bejan [14].

## 8. Results and discussion

No experimental data are available to this point to reliably estimate values of dimensionless parameters characterizing the behavior of oxytactic microorganisms in a porous medium. Since the aim of this paper is to investigate trends,  $\varpi = Pe\beta = 1$  and  $\delta = 1$  are adopted for the basic case, and then values of  $\varpi$  and  $\delta$  are varied around unity in Figs. 2 and 3.

Fig. 2 displays the effect of the Rayleigh number,  $\hat{Ra}$ , on the critical value of the bioconvection Rayleigh number,  $\hat{Rb}_{cr}$ , and on the critical wavenumber,  $a_{cr}$ . Fig. 2 is computed for three fixed values of  $\varpi$ : 0.1, 1, and 10. The decrease of  $\hat{Rb}_{cr}$  with increase of  $\hat{Ra}$  means that increasing the temperature variation across the porous layer destabilizes the suspension and helps the development of bioconvection. As expected, the curves in Fig. 2(a) intersect at the point  $(\hat{Ra}, \hat{Rb}) = (40, 0)$ . The curves in Fig. 2(b) intersect at the point  $(\hat{Ra}, a) = (40, 3.16)$ . When  $\hat{Ra}$  exceeds 40, convection

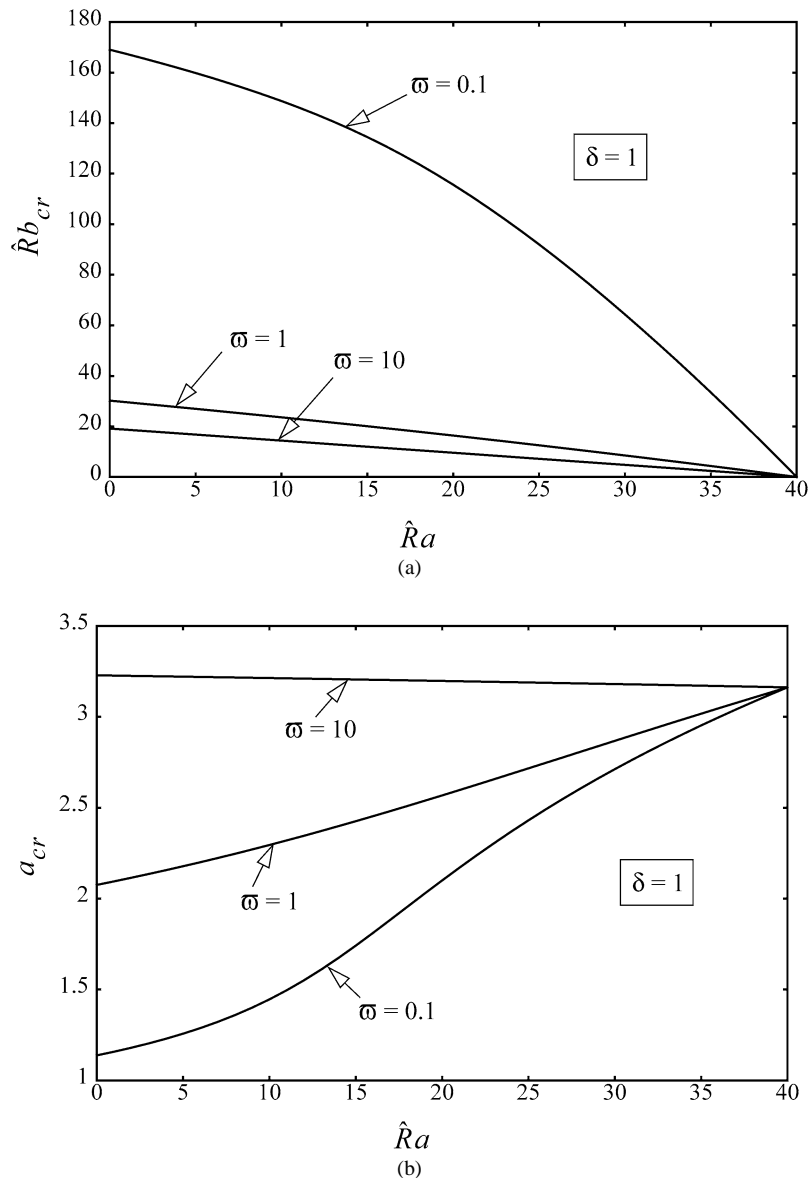


Fig. 2. Effect of the traditional “thermal” Rayleigh number,  $\hat{Ra}$ , on the critical value of the bioconvection Rayleigh number,  $\hat{Rb}_{cr}$  (a) and on the critical horizontal wavenumber,  $a_{cr}$  (b). Different curves correspond to different fixed values of  $\varpi$ .



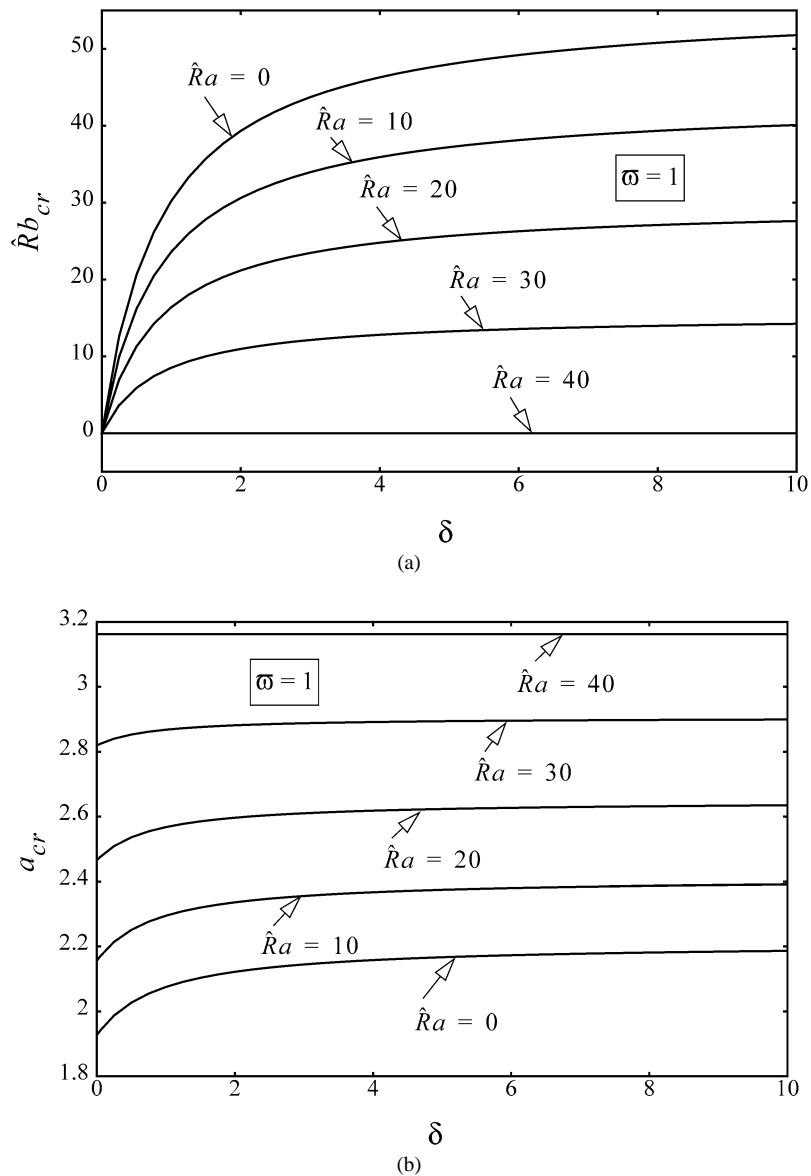


Fig. 3. Effect of  $\delta$  on the critical value of the bioconvection Rayleigh number,  $\hat{Rb}_{cr}$  (a) and on the critical wavenumber,  $a_{cr}$  (b). Different curves correspond to different fixed values of  $\hat{Ra}$ .

develops even in a suspension of buoyancy-neutral microorganisms because of the unstable density stratification caused by heating the porous layer from below.

Hillesdon and Pedley [2] interpreted  $\varpi$  as a depth parameter; larger values of  $\varpi$  correspond to a steeper free-surface density gradient in the basic state. It is expected that a steeper free-surface density gradient leads to a more unstable system, which explains the decrease of  $\hat{Rb}_{cr}$  with the increase of  $\varpi$ , as observed in Fig. 2(a).

Fig. 3 displays the effect of  $\delta$  on  $\hat{Rb}_{cr}$  and  $a_{cr}$ . This figure is computed for five fixed values of  $\hat{Ra}$ : 0, 10, 20, 30, and 40.  $\delta$  characterizes the ratio of oxygen diffusivity to that of microorganisms (the diffusivity of microorganisms is mostly due to the random component in their swimming). From Fig. 3(a) it is clear that  $\hat{Rb}_{cr}$  first increases rapidly as  $\delta$  increases, but then approaches an asymptotic limit. It should be noted that in a shallow chamber (considered in this paper) the oxygen and cell concentrations in the basic state (given by Eqs. (22) and (23) respectively) depend on  $\varpi$  but do not depend on  $\delta$ . The increase of  $\hat{Rb}_{cr}$  when increasing  $\delta$  is consistent with the findings of Hillesdon and Pedley

[2] for small values of  $\delta$  (see Fig. 7(a) in their paper); however, the maximum that the dependence  $\widehat{Rb}_{cr}(\delta)$  takes on at a certain value of  $\delta$  is not observed in Fig. 3(a); this may be due to what the porous medium brings to the problem.

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## Appendix A

Functions  $I_1$ – $I_5$  in Eq. (60) for the critical bioconvection Rayleigh number are given below:

$$I_1(\varpi) = A_1 \int_0^1 (2-z)(-1+z)z^2 \tan\left(\frac{1}{2}A_1(1-z)\right) dz, \quad (\text{A.1})$$

$$\begin{aligned} I_2(\varpi) = & \frac{1}{4}A_1\varpi \int_0^1 F(z, \varpi) \\ & \times \left[ A_1 \sec^2\left(\frac{A_1}{2}\right) \sec^2\left(\frac{1}{2}A_1(-1+z)\right) (2 - A_1^2(-2+z)z + 2\cos(A_1) + A_1(-2+z)z \sin(A_1)) \right. \\ & \left. + \frac{8(A_1 - \sin(A_1))(1 - A_1(-1+z) \tan(\frac{1}{2}A_1(-1+z)))}{1 + \cos(A_1)} \right] dz, \end{aligned} \quad (\text{A.2})$$

$$I_3(\varpi) = A_1^2\varpi \int_0^1 \sec^2\left(\frac{1}{2}A_1(-1+z)\right) F(z, \varpi) \left(-1 - A_1(-1+z) \tan\left(\frac{1}{2}A_1(-1+z)\right)\right) dz, \quad (\text{A.3})$$

$$I_4(\varpi) = \int_0^1 (-2+z)z \sec^2\left(\frac{1}{2}A_1(-1+z)\right) F(z, \varpi) dz, \quad (\text{A.4})$$

$$\begin{aligned} I_5(\varpi) = & -A_1^3 \int_0^1 (-1+z)z \sec^2\left(\frac{1}{2}A_1(-1+z)\right) \left(1 + \frac{A_1(z - 0.5z^2)(A_1 - \sin(A_1))}{1 + \cos(A_1)}\right) \\ & \times \tan\left(\frac{1}{2}A_1(-1+z)\right) dz \end{aligned} \quad (\text{A.5})$$

where

$$F(z, \varpi) = 1 + \frac{1}{4}A_1(-2+z)z \sec^2\left(\frac{A_1}{2}\right) (-A_1 + \sin(A_1)). \quad (\text{A.6})$$

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